ROYAL MCBEE CORPORATION
ELECTRONIC COMPUTER DEPARTMENT

INTEGRATION OF DIFFERENTIAL EQUATIONS
(Gill's Method)

(Program 28.0)

FUNCTION:

To integrate n first-order differential equations of the form

$$\frac{dy^j}{dt} = f^j\,(t, y^1, y^2, \ldots, y^n). \quad *$$

The method is described in detail at the end. (The variables are designated by superscripts because subscripts are reserved for something else.)

INPUT:

Provided in the initial set-up part of the main program, which is used only at the very beginning, and in the calling sequence.

OUTPUT:

The new values of the functions, and of other quantities carried along to make the calculations possible (see discussion of method).

For each variable must be stored four quantities:

y^j;

$f^j = \frac{dy^j}{dt}$;

$\hat{q}^j$, a quantity carried along only for use in the computation;

and $m^j = \begin{cases} 2(q_f j - q_y j) & \text{if } q_f j < q_y j \\ \text{any negative number} & \text{if } q_f j = q_y j \end{cases}$

$q_f j > q_y j$ is forbidden.

$q_f j$ is the q of the jth derivative;

$q_y j$ is the q of the jth function.

Note again that the derivative may not be carried at a higher q than the corresponding function.

It is assumed that the functions and derivatives are all carried at the lowest possible q's, subject to the restriction of the preceding paragraph.

* A later section deals with the problem of integrating equations of order larger than 1.

STORAGE:

The routine itself occupies 209 cells, $(L_0 - L_0 + 0208)_{10}$, but L_0+17-19, L_0+57, L_0+125, L_0+153, L_0+161-3 are not used; it also uses 11 cells on track 63:

6305, 6311, 6318, 6339-41, 6345, 6355-6, 6358, 6362.

In addition, four quantities must be stored for each variable.

Let the n variables be stored in $Y^1 - Y^n$;

the n derivatives in $F^1 - F^n$;

the n "modifiers" in $M^1 - M^n$;

The n q's in $Q^1 - Q^n$.

It is seen that, in general, 4n storages must be set aside for the functions and the quantities necessary to compute them. *

In the following discussion h is the interval of integration i.e.,

$$h = t_{k+1} - t_k.$$

CALLING SEQUENCE:

Actually, two calling sequences are necessary. In the "initial set-up" portion of the program, used only at the very beginning, must appear:

Location	Order	Address	
β - 2	B	L(C)	C is usually B $(L_o+62)_{10}$**
β - 1	H	$(L_o+140)_{10}$	C ⟶ L_o+140
β	B	L(h)	H at 0 (h=1 has the hex. representation 7wwwwwwq.)
β + 1	H	$(L_o+0154)_{10}$	⟶ L_o+154
β + 2	B	$(L_o+0160)_{10}$	
β + 3	C	$(L_o+0038)_{10}$	0 ⟶ AC
β + 4	H	(Q_1)	
β + 5	H	(Q_2)	$\hat{q}^j$ made for all j.
.	.		
.	.		
.	.		
β + (n+3)	H	(Q_n)	

* The exception, in integrating n-th-Order equations, is discussed later.
** In one special case this command will be something else. (See page 6.)

β and β+1 store the interval of integration. β+2 and β+3 initialize the routine correctly. β+4 through β+(n+3) make the n $\hat{q}$'s 0. The $\hat{q}^j$ must be made 0 at the start and at any time new initial conditions are read in. The rest of the time they are stored by the routine itself, along with the computed values of the y^j. The purpose of β+4 through β+(n+3) may, of course, be served by a loop or by data input.

The independent variable must be carried as the solution of a differential equation.

$$\frac{dz}{dt} = 1, \quad z(t_o) = z_o.$$

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ELECTRONIC COMPUTER DEPARTMENT

INTEGRATION OF DIFFERENTIAL EQUATIONS (Radioplane)
(Program 28.0)

The calling sequence in the main program is:

Location	Order	Address	
α - 15	B	L[(n-2) at 29]	(n-2) at 29
	H	$(L_o+0102)_{10}$	$\longrightarrow$ 0102
	B	$L(M^1)$	$M^1 \longrightarrow$
	Y	$(L_o+0203)_{10}$	$A(L_o+0203)$
	B	$L(Y^1)$	$Y^1 \longrightarrow$
	Y	$(L_o+0130)_{10}$	$A(L_o+0130)$ &
	Y	$(L_o+0131)_{10}$	$A(L_o+0131)$
	B	$L(F^1)$	$F^1 \longrightarrow$
	Y	$(L_o+0036)_{10}$	L_o+36
	B	$L(Q^1)$	$Q^1 \longrightarrow$
	Y	$(L_o+0126)_{10}$	$A(L_o+0126)$ &
	Y	$(L_o+0200)_{10}$	$A(L_o+200)$
α - 3	U	$(\alpha -1)$	
	U	---- Special return, for i = 4	
	R	$(L_o+0101)_{10}$	
	U	$(L_o+0200)_{10}$	
α + 1		---- ordinary return, for i< 4	

The return will be to α-2 or to α+1. Since this is, essentially, a Runge-Kutta scheme, there must be 2 different returns. Only values at i = 4 "count", so the "end logic" (testing, printing, etc.) will be done only when i = 4. α +1, therefore, will usually be a return to the section of the program in which the derivatives are evaluated.

Notes: It might be thought that $h \leq 1$ is a severe restriction; actually, it is not. One can easily make a change of variable to get the new interval down to size, <u>and change his equations accordingly</u>.

It is very easy to integrate mth-order equations ($m > 1$) with this scheme. Suppose

$$\frac{d^m y}{dt^m} = F(t, y^{(m-1)}, y^{(m-2)}, ---, y), \text{ where}$$

$$y^{(j)} = \frac{d^j y}{dt^j} .$$

Then,
$$y^{(1)} = f^1;$$
$$y^{(2)} = f^2 = \frac{d}{dt} f^1;$$
$$\vdots$$
$$y^{(m)} = f^m = \frac{d}{dt} f^{(m-1)}$$

Incidentally, if one stores y, $y^{(1)}$, ... $y^{(m)}$ consecutively, m + 1 cells will suffice to hold y and its first m derivatives instead of 2m cells. (Therefore, $F^1 = Y^1 + 1$.) Furthermore, $y^{(j)}$, $1 \leq j \leq m - 1$, need be stored only once, not twice.

ACCURACY:

Not easy to estimate, but the developer of this method puts the upper bound of the error at

$$-\frac{1}{120} h^5 y$$ per time step, or

$$-\frac{1}{120} h^5 \Sigma y$$ over the whole range.

TIME:

Roughly $^{4}/_{3}n - {}^{1}/_{5}$ sec. / variable (based on 15.2 msec. / revolution). + 4 times the time to evaluate the derivatives.

DESCRIPTION OF METHOD:

The procedure used is S. Gill's modification of the fourth-order Runge-Kutta method, with two changes to enable greater accuracy than provided by equations (26) of Gill's paper.*

For each of n variables must be solved a first-order differential equation

$$\frac{dy^j}{dt} = f^j (y^1, y^2, \ldots, y^n, t).$$

The last equations show what the $\hat{q}^j$ do; it is not yet clear what purposes the modifiers, $m^j = 2qf^j - qy^j$, serve. h is carried at 0. In (1c), (2c), (3c), and (4c), we have (without superscripts).

$$(5)\ y_i = y_{i-1} + hr\hat{}_i.$$

*S. Gill, "A Process for the Step-by-step Integration of Differential Equations in an Automatic Digital Computing Machine." Proceedings Cambridge Philosophical Society. Vol 47, Pt. 1.

The $\hat{r_i}$ are linear combinations of derivatives, and the derivatives and corresponding functions may be at different q's. In the machine, then, (5) is calculated by (6):

$$\text{(6)}\quad y_i \text{ at } q_y = y_{i-1} \text{ at } q_y + (h \text{ at } 0)\ (1 \text{ at } q_y - q_f)\ (\hat{r_1} \text{ at } q_f)$$

The 1 at $(q_y - q_f)$ is $2^{q_f - q_y}$, or the modifier previously defined.

Note: It is possible to save (n-1) more cells (and a little time in the integration routine) in one special case. Suppose

$$q_{fj} - q_{yj} = \text{constant for all } j.$$

By far the commonest situation in which this relation would occur is that in which

$$q_{fj} = q_{yj} \text{ for all } j.$$

In this case only, only one modifier (in M^1) is necessary, instead of n modifiers (in M^1-M^n). Then, one command must be changed and the "preliminary" calling sequence must read:

Location	Order	Address	
β - 2	B	L(C)	In this case, C = U (L_o + 143)
	H	$(L_o + 140)_{10}$	C $\longrightarrow$ L_o + 140
β	B	L(h)	

.
.
.

As used here, the equations for each variable in turn are:

1st step,

$$i = 1 \quad \begin{cases} \text{(1a)}\ \hat{r_1} = 1/2\ f_o - \hat{q_o} \\ \text{(1b)}\ \hat{q_1} = \hat{q_o} + 3\hat{r_i} - 1/2\ f_o \\ \text{(1c)}\ y_1 = y_o + h\ \hat{r_1} \end{cases}$$

2nd step,

$$i = 2 \quad \begin{cases} \text{(2a)}\ \hat{r_2} = (1 - \sqrt{1/2})(f_1 - \hat{q_1}) \\ \text{(2b)}\ \hat{q_2} = \hat{q_1} + 3\hat{r_2} - (1 - \sqrt{1/2})(f_1 - \hat{q_1}) \\ \text{(2c)}\ y_2 = y_1 + h\ \hat{r_2} \end{cases}$$

3rd step,

$$i = 3 \quad \begin{cases} \text{(3a)}\ \hat{r_3} = (1 + \sqrt{1/2})(f_2 - \hat{q_2}) \\ \text{(3b)}\ \hat{q_3} = \hat{q_2} + 3\hat{r_3} - (1 + \sqrt{1/2})\ f_2 \\ \text{(3c)}\ y_3 = y_2 + h\hat{r_3} \end{cases}$$

4th and last step,

$$i = 4 \quad \begin{cases} (4a)\ \hat{r_4} = 1/6\ f_3 - 1/3\ \hat{q_3} \\ (4b)\ \hat{q_4} = \hat{q_3} + 3\hat{r_4} - 1/2\ f_3 \\ (4c)\ y_4 = y_3 + h\ \hat{r_4} \end{cases}$$

One can check out the evaluation of his derivatives by printing the q^j. The following equations are appended for this purpose.

$$\begin{cases} \hat{r_1} \approx 1/2\ f_o; \\ \hat{q_1} \approx f_o. \end{cases}$$

$$\begin{cases} \hat{r_2} \sim 0; \\ \hat{q_2} \sim \sqrt{1/2}\ f_o. \end{cases}$$

$$\begin{cases} \hat{r_3} \sim 1/2\ f_o; \\ \hat{q_3} \sim 1/2\ f_o. \end{cases}$$

$$\begin{cases} \hat{r_4} \sim 0; \\ \hat{q_4} \sim 0. \end{cases}$$

If y is linear, the "$\sim$" and "$\approx$" would be "="; $\hat{q_4}$ should always be identically 0, and will be except for round-off.

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A TYPICAL PROBLEM FOR USE WITH SUBROUTINE No. 28.0

STATEMENT OF THE PROBLEM: "Integrate the following system of differential equations with the given initial conditions:

EQUATIONS: $\frac{dx}{dt} = y$; $\frac{dy}{dt} = -x$; $\frac{dz}{dt} = 2$

INITIAL CONDITIONS: $x = 0$; $y = 1$; $z = 0$.

Since the results of the process of numerical integration will give the values of the functions or variables, "x","y" & "z" at the end of a specified time interval, an alternative statement of the problem would run as follows:

"Given the system of differential equations

$$\frac{dx}{dt} = y \quad ; \quad \frac{dy}{dt} = -x \quad ; \quad \frac{dz}{dt} = 2$$

and the initial values of the functions at the time $t = 0$, namely,

$$x = 0 \quad ; \quad y = 1 \quad ; \quad z = 0 \quad ,$$

compute the values of the functions at the time $t = 1/8$.

The punched tape for this problem contains linkage instructions or "calling" sequences for three entities:

I. The problem's "data"
   a) functions
   b) derivatives
   c) modifiers
   d) special constants " $\hat{q}$ "

II. Subroutine No. 28.0 (Stored on tracks 50, 51, and 52)

III. Data Output No. 1 (Stored on tracks 08, and 09.)

The names, initial values, and corresponding scale factors during storage and output printing are as follows:

QUANTITY	In. VALUE	STORAGE "Q"	OUTPUT "Q"	LOCATION
1. x	0	1	4	1003
2. y	1	1	4	1002
3. z	0	7	7	1001
4. dx/dt	0	1	4	1035
5. dy/dt	0	1	4	1034
6. dz/dt	2	2	4	1033
7. $\hat{q}_x$	1	1	4	5620
8. $\hat{q}_y$		1	4	5619
9. $\hat{q}_z$		2	4	5618

The operating program may be adjusted, via three instructions, to print the above nine quantities at the end of each stage "i", or at the end of a given time interval "h", or at the end of a multiple of "h" depending on the words in the locations: 2024, 2027, and 6234.

A TYPICAL PROBLEM FOR USE WITH SUBROUTINE NO. 28.0

These three printing possibilities may be tabularly listed with their locations and corresponding instructions as follows:

PRINTING AT THE END OF :

EACH "i"		EACH TIME INTERVAL "h"**		A MULTIPLE OF "h"	
Location	Instruction	Location	Instruction	Location	Instruction
2024 :	U 6200	2024 :	U 6200	2024 :	U 1920
2027 :	U 6200	2027 :	U 1906	2027 :	U 1906
6234 :	U 1906	6234 :	U 1936	6234 :	U 1936

Automatic carriage return key is to be placed such that automatic return of the carriage follows the printing of the ninth information item, "q_z". (The "q's" with the circumflex symbols are special calculational constants and are not related to scaling processes.)

As can be seen from printing the contents of the problem tape the operating program, or control program, is stored in locations on tracks 17, 19, 20, and 62. This particular storage arrangement is completely arbitrary.

A copy of the program description for Subroutine No. 28.0 should be available to the reader when using this particular typical problem.

PROCEDURE

1. Store Data Output beginning on 0800
2. Store "A Program for Operating the "Integration of Differential Equations" Subroutine No. 28.0"; this tape has all of the necessary "Start Fill" Set Mod" information all punched on tape.
3. Store Subroutine No. 28.0 beginning in location 5000.
4. Execute a transfer to location 1710 via ".0001710' ".
5. When this is done, the program will compute the nine pertinent values and will print them out for time equal to 1/8 second, as the tape is presently punched. To obtain other intervals of printing listed above, make changes in words indicated.

** As the problem tape is presently punched it will execute printing at the end of each time interval "h".

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Job No. 92 Prog. No. 28.0 Prep. by E.M. Stone Ck'd by M. Moore Date

Problem: Integration of Simultaneous Differential Equations (Radioplane) Track

Program Input Codes	Stop	Location	Op.	Address	Stop	Contents of Address	Notes
	'						
	'	⊠					
		0000			'	$h[2^{q_x - q_y}]$ $(q_x \leq q_y)$	
0000 0001		01		4	'	1@29	
		02	XH	6345	'		
		03	M	0100	'	⊠ $\frac{1}{2}$	
		04	XH	6340	'		
		05	XS	6362	'	$\hat{q}_i$ ($\hat{q}_0$, in this case)	
		06	XH	6358	'		
		07	XA	6362	'	⊠ $\hat{q}_0$	
		08	XS	6340	'	$\frac{1}{2}\hat{k}_0$	
		09	XA	6358	'	$\hat{r}_i$ ($\hat{r}_0$, in this case)	
		10	XA	6358	'	$\hat{r}_1$	
		11	R	0133	'	⊠	
		12	U	0126	'		
		13	U	0014	'		
		14	B	0157	'	U0020	
		15	H	0038	'	⊠	
		16	U	0101	'		go to exit
		17			'		
		18			'		
		19			'	⊠	
		20	XH	6356	'		
		21	XS	6362	'	$\hat{q}_1$	
		22	M	0058	'	$1-\sqrt{\frac{1}{2}} = C_1$	
		23	XH	6358	'	⊠	
		24	D	0060	'	$\frac{1}{3}$	
		25	XA	6362	'	$\hat{q}_1$	
		26	XC	6305	'		
		27	XS	6356	'	⊠ $\hat{k}_1$	
		28	M	0058	'	$1-\sqrt{\frac{1}{2}} = C_1$	
		29	XA	6305	'	$\hat{q}_1 + 3\hat{r}_2$	
		30	R	0133	'		
		31	U	0126	'	⊠	

' Conditional Stop Code ⊠ Carriage Return

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Job No. ____ Prog. No. 28.0 Prep. by E. M. Stone Ck'd. by M. Moore Date ____

Problem Integration of Simultaneous Differential Equations (Radioplane) Track ____

Program Input Codes	Stop	Location	Instruction Op. Address	Stop	Contents of Address	Notes
	'					
	'	⊠				
		0032	U 0033	'		
		33	B 0158	'	U0039	
		34	H 0038	'		
		35	U 0101	'	⊠	to exit
		36	B [ ]	'		
		37	U 0038	'		
		38	U [0002]	'		
		39	XH 6311	'	⊠	
		40	XS 6362	'	$\hat{q}_2$	
		41	M 0063	'	$\frac{1}{2}c_2 = \frac{1}{2}(1+\sqrt{\frac{1}{2}})$	
		42	D 0100	'	$\frac{1}{2}$	
		43	XC 6358	'	⊠	
		44	XS 6311	'	$\hat{k}_2$	
		45	M 0063	'	$\frac{1}{2}[c_2 = 1+\sqrt{\frac{1}{2}}]$	
		46	XH 6318	'		
		47	XA 6362	'	⊠ $\hat{q}_2$	
		48	XA 6358	'	$\hat{k}_3$	
		49	XA 6318	'	$-\frac{1}{2}c_2\hat{k}_2$	
		50	XA 6358	'	$\hat{k}_3$	
		51	XA 6358	'	⊠ $\hat{k}_3$	
		52	R 0133	'		
		53	U 0126	'		(
		54	B 0159	'	U0103	
		55	H 0038	'	⊠	
		56	U 0101	'		to exit
		57		'		
0000 0007		58	257K 8666	'	$c_1 = 1-\sqrt{\frac{1}{2}}$	
		59	1555 5556	'	⊠ $\frac{1}{6}$	
		60	2FFF FFFF	'	$\frac{1}{3}$	
		61	J	'	3 @ 29	
		62	4	'	1 @ 29	
		63	6K41 3JJJ	'	⊠ $\frac{1}{2}(c_2 = 1+\sqrt{\frac{1}{2}})$	

' Conditional Stop Code ⊠ Carriage Return

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PREPARED FOR:				PAGE 3 OF 5
JOB NO.	PROGRAM NO. 28.0	PROGRAM PREPARED BY: E.M. Stone	PROGRAM CHECKED BY: M. Moore	DATE
PROBLEM: Integration of Simultaneous Differential Equations (Radioplane)				TRACK

PROGRAM INPUT CODES	STOP	LOCATION	INSTRUCTION OPERATION / ADDRESS	STOP	CONTENTS OF ADDRESS	NOTES
	/					
	/	☒				
		0100	40090000	/	$\frac{1}{2}$	
		0101	U[ ]	/		exit
		0102	[ ]	/	counter (n-2) @ 29	
		0103	XH6339	/	☒	
		0104	M0059	/	$\frac{1}{6}$	
		0105	XC6341	/		
		0106	XS6362	/	$\hat{q}_3$	
		0107	M0060	/	☒ $\frac{1}{3}$	
		0108	XA6341	/	$\frac{1}{2}\hat{k}_3$	
		0109	XH6358	/		
		0110	D0060	/	$\frac{1}{3}$	
		0111	XA6362	/	☒ $\hat{q}_3$	
		0112	XC6355	/		
		0113	S0106	/	$\frac{1}{2}$	
		0114	XM6339	/		
		0115	XA6355	/	☒ $\hat{q}_3 + 3\hat{k}_4$	
		0116	R0133	/		
		0117	U0126	/		
		0118	U0119	/		
		0119	B0160	/	☒	
		0120	H0038	/		
		0121	B0101	/	U(α+1)	
		0122	S0061	/	3@29	
		0123	H0101	/	☒	
		0124	U0101	/		to exit
		0125		/		
		0126	H[ ]	/		
		0127	U0128	/	☒	
		0128	B0000	/	$\frac{1}{2}[2q + \hat{q}_3]$	
		0129	XM6358	/	$\frac{1}{2}\hat{k}$	
		0130	A[ ]	/		
		0131	H[ ]	/	☒	

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Job No. ______ Prog. No. 28.0 Prep. by E. M. Stone Ck'd. by M. Moore Date ______

Problem Integration of Simultaneous Diff. Equations (Radioplane) Track ______

Program Input Codes	Stop	Location	Instruction Op. Address	Stop	Contents of Address	Notes
	'					
	'	⊠				
		0132	B0102	'	(n-2)@29	
		33	T[ ]	'		
		34	S0001	'	1@29	
		35	H0102	'	⊠	
		36	B0126	'		
		37	A0062	'	1@29	
		38	Y0200	'		
		39	H0126	'	⊠	
		40	B0062	'	1@29	
		41	A0103	'		
		42	H0203	'		
		43	B0062	'	⊠ 1@29	
		44	A0130	'	A{L(y^J)}	
		45	Y0131	'		
		46	Y0130	'		
		47	B0062	'	⊠ 1@29	
		48	B0036	'		
		49	U0150	'		
		50	H0036	'		
		51	U0152	'	⊠	
		52	U0200	'		
00000002		53	28W5J28	'	0.02	
		54	[ ]	'	h	
		55	B0154	'	⊠ h	
		56	U0206	'		
		57	U0020	'		
		58	U0039	'		
		59	U0103	'	⊠	
		60	U0002	'		
00000003		61	147FQ14	'	0.01	
		62	JJJJJ33	'	0.1	
		63	199999F	'	⊠ 0.2	

' Conditional Stop Code ⊠ Carriage Return

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Job No. ______ Prog. No. 28.0 Prep. by E.N. Stone Ck'd. by M. Moore Date ______

Problem Integration of Simultaneous Differential Equations (Radioplane) Track ______

Program Input Codes	Stop	Location	Instruction Op.	Address	Stop	Contents of Address	Notes
	'						
	'	☒					
		0200	B	[ ]	'	$\hat{q}(j)$	
		01	XH	6362	'		
		02	U	0203	'		
		03	B	[ ]	'	☒ $2 q_f(j) - q_y(j)$ or neg. #	
		04	T	0155	'		
		05	M	0154	'	h	
		06	H	0000	'		
		07	U	0208	'	☒	
		08	U	0036	'		
		09			'		
		10			'		
		11			'	☒	
		12			'		
		13			'		
		14			'		
		15			'	☒	
		16			'		
		17			'		
		18			'		
		19			'	☒	
		20			'		
		21			'		
		22			'		
		23			'	☒	
		24			'		
		25			'		
		26			'		
		27			'	☒	
		28			'		
		29			'		
		30			'		
		31			'	☒	

' Conditional Stop Code ☒ Carriage Return